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Quantum Gates and Quantum Algorithms with Clifford Algebra Technique

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Abstract We use the Clifford algebra technique (J. Math. Phys. 43:5782, 2002; J. Math. Phys. 44:4817, 2003), that is nilpotents and projectors which are binomials of the Clifford algebra objects γ^a with the property $\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}$, for representing quantum gates and quantum algorithms needed in quantum computers in a simple and an elegant way. We identify n-qubits with the spinor representations of the group $SO(1,3)$ for a system of n spinors. Representations are expressed in terms of products of projectors and nilpotents; we pay attention also on the nonrelativistic limit. An algorithm for extracting a particular information out of a general superposition of 2^n qubit states is presented. It reproduces for a particular choice of the initial state the Grover's algorithm (Proc. 28th Annual ACM Symp. Theory Comput. 212, 1996).

Keywords Clifford algebra · Quantum gates · Quantum algorithms · Group $SO(1,3)$

1 Introduction

In the references [4, 5] and also in the references cited there the use of the geometrical algebra to demonstrate the gates and their functioning is presented. In this paper we use the technique from the references [1, 2], which represents spinor representations of the group $SO(1,3)$ in terms of projectors and nilpotents—the binomials of the Clifford algebra objects γ^a. We treat relativistic and nonrelativistic cases although at this moment the experimentally realizable quantum computers do not need the relativistic treatment.[1] Fast electrons in the ring, for example, used to function as a quantum computer, would, however, need the relativistic treatment. We demonstrate that in the nonrelativistic limit the group $SO(1,3)$ is behind the usually used group SU(2).

[1] The reader can find in the Ref. [6] and the references cited there discussions about the present status of quantum computers, while we recommend the Ref. [7] to learn more about the Clifford algebra. One can find the appropriate references about the Clifford algebra also in [1, 2].

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It is easy to prove (and it is also well known) that any type of a quantum gate, operating on one qubit and represented by an unitary operator, can be expressed as a product of the two types of quantum gates—the phase gate and the Hadamard's gate—while the C-NOT gate, operating on two quantum bits, enables to make a quantum computer realizable, since all needed operations can be expressed in terms of these three types of gates. In this paper we identify the spinor representation of two one spinor states with the two quantum bits $|0\rangle$ and $|1\rangle$ and accordingly n spinors' representation of $SO(1,3)$ with the n-qubits. The three types of the gates can then be expressed in terms of projectors and nilpotents in a transparent and elegant way.

We present also the algorithm for extracting a particular information out of any superposition of a n-qubit state. For a particular choice of the initial n-qubit state this general algorithm reproduces the Grover's algorithm [3].

2 The Technique for Spinor Representations and Qubits

We define in this section basic states for the spinor representation of the group $SO(1,3)$ using the technique of the Refs. [1, 2] and identify the two qubits with the two spinor states of a chosen either handedness or parity. The n-qubits are then identified with the n two spinor states.

There are twice 2 basic states of the spinor (fundamental) representation of the group $SO(1,3)$, two $(2^{4/2-1})$ of each handedness (the left one and the right one). We represent the basic states as polynomials of the Clifford algebra objects γ^a—nilpotents and projectors [1, 2]—chosen to be the eigenvectors of the Cartan subalgebra set of the $2(=\frac{4}{2})$ commuting operators (we make the usual choice of the Cartan subalgebra set: S^{12} and S^{03}) out of the set of $6 = (4(4-1)/2)$ infinitesimal operators S^{ab} (S^{01}, S^{02}, S^{03}, S^{23}, S^{31}, S^{12}) of the group $SO(1,3)$, fulfilling the Lorentz algebra $\{S^{ab}, S^{cd}\}_- = i(\eta^{ad}S^{bc} + \eta^{bc}S^{ad} - \eta^{ac}S^{bd} - \eta^{bd}S^{ac})$. The generators S^{ab} can for spinors be written in terms of the operators γ^a fulfilling the Clifford algebra

$$\begin{aligned} &\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab}, \quad \mathrm{diag}(\eta) = (1,-1,-1,-1), \\ &S^{ab} = \frac{i}{2}\gamma^a\gamma^b, \quad \text{for } a \neq b \text{ and } 0 \text{ otherwise.} \end{aligned} \tag{1}$$

They define the spinor (fundamental) representation of the group $SO(1,3)$ of the two handedness, the left (L) and the right (R) one

$$\begin{aligned} |0\rangle_L &= \overset{03}{[-i]}\overset{12}{(+)}\,|\varphi_0\rangle, \quad |1\rangle_L = \overset{03}{(+i)}\overset{12}{[-]}\,|\varphi_0\rangle, \\ |0\rangle_R &= \overset{03}{(+i)}\overset{12}{(+)}\,|\varphi_0\rangle, \quad |1\rangle_R = \overset{03}{[-i]}\overset{12}{[-]}\,|\varphi_0\rangle, \end{aligned} \tag{2}$$

with nilpotents ($\overset{ab}{(k)}\overset{ab}{(k)}= 0$) and projectors ($\overset{ab}{[k]}\overset{ab}{[k]}=\overset{ab}{[k]}$) defined as

$$\begin{aligned} \overset{03}{(\pm i)} &:= \frac{1}{2}(\gamma^0 \mp \gamma^3), \quad \overset{12}{(\pm)} := \frac{1}{2}(\gamma^1 \pm i\gamma^2), \\ \overset{03}{[\pm i]} &:= \frac{1}{2}(1 \pm \gamma^0\gamma^3), \quad \overset{12}{[\pm]} := \frac{1}{2}(1 \pm i\gamma^1\gamma^2). \end{aligned} \tag{3}$$

Nilpotents and projectors are eigenstates of the Cartan subalgebra set S^{03} and S^{12}, since $S^{03}\stackrel{03}{(\pm i)} = \pm\frac{i}{2}\stackrel{03}{(\pm i)}$, $S^{03}\stackrel{03}{[\pm i]} = \pm\frac{i}{2}\stackrel{03}{[\pm i]}$ and similarly $S^{12}\stackrel{12}{(\pm)} = \pm\frac{1}{2}\stackrel{12}{(\pm)}$, $S^{12}\stackrel{12}{[\pm]} = \pm\frac{1}{2}\stackrel{12}{[\pm]}$. This can very easily be checked just by applying S^{03} and S^{12} on a particular nilpotent $(\stackrel{03}{(\pm i)}, \stackrel{12}{(\pm)})$ or projector $(\stackrel{03}{[\pm i]}, \stackrel{12}{[\pm]})$ and by using (1). $|\varphi_0\rangle$ in (3) is the (in our technique unimportant) vacuum state [1, 2] (which can be for our choice of basic states represented by $|\varphi_0\rangle = \stackrel{03}{[-i]}\stackrel{12}{[-]}\,|0\rangle$ and normalized as $\langle\varphi_0|\varphi_0\rangle = 1$), which does not influence our derivations and we shall from now on skip it.

The states $|0\rangle_L$ and $|1\rangle_L$ have the handedness $\Gamma = -4iS^{03}S^{12}$ equal to -1, while the states $|0\rangle_R$ and $|1\rangle_R$ have the handedness equal to 1, which again can easily be proved just by inspection. When taking into account that $\gamma^{a\dagger} = \eta^{aa}\gamma^a$ it follows that $(\stackrel{03}{(\pm i)})^\dagger = \stackrel{03}{(\mp i)}$, $(\stackrel{12}{(\pm)})^\dagger = -\stackrel{12}{(\mp)}$, $(\stackrel{03}{[\pm i]})^\dagger = \stackrel{03}{[\pm i]}$ and $(\stackrel{12}{[\pm]})^\dagger = \stackrel{12}{[\pm]}$. It can then easily be checked (by taking into account the normalization of the vacuum state $\langle\varphi_0|\varphi_0\rangle = 1$ and (1)) that the basic spinor states are normalized as follows [1, 2]

$$
{}_\beta\langle i|j\rangle_\alpha = \delta_{ij}\delta_{\alpha\beta}, \tag{4}
$$

where i, j denote 0 or 1 and α, β left or right handedness.

When describing a spinor in its center of mass motion, the representation with a well defined parity is usually more convenient

$$
\begin{aligned}
|0\rangle_\pm &= \frac{1}{\sqrt{2}}(\stackrel{03}{[-i]}\stackrel{12}{(+)} \pm \stackrel{03}{(+i)}\stackrel{12}{(+)}),\\
|1\rangle_\pm &= \frac{1}{\sqrt{2}}(\stackrel{03}{(+i)}\stackrel{12}{[-]} \pm \stackrel{03}{[-i]}\stackrel{12}{[-]}),
\end{aligned} \tag{5}
$$

where $+, -$ stay for the positive and the negative parity state, respectively. Both representations, the chiral one and the one of the well defined parity, are equivalent.[2]

Let us present some useful relations for the Clifford algebra objects. Nilpotents and projectors fulfil the following relations [1, 2] (which can be checked just by using the definition of the nilpotents and projectors (3) and by taking into account the Clifford property of

[2] It is worthwhile to notice that there are the two superposition of $\stackrel{03}{(+i)}$ and $\stackrel{03}{[-i]}$ (namely $\frac{1}{\sqrt{2}}(\stackrel{03}{(+i)} \pm \stackrel{03}{[-i]})$), which are eigenstates of γ^0 (with the eigenvalue ± 1, respectively) and define the positive parity state (with the $+$ sign) and the negative parity state (with the $-$ sign), while $\stackrel{12}{(+)}$ or $\stackrel{12}{[-]}$ determines the eigenstates of S^{12}. Let us point out that any operator $S^{ij}, i = 1, 2, 3;$ transforms a left handed basic state into a left handed basic state and a right handed basic state into a right handed basic state. The operator S^{31}, for example, transforms the state $\stackrel{03}{[-i]}\stackrel{12}{(+)}$ into $\frac{i}{2}\stackrel{03}{(+i)}\stackrel{12}{[-]}$, while it transforms $\stackrel{03}{(+i)}\stackrel{12}{(+)}$ into $\frac{i}{2}\stackrel{03}{[-i]}\stackrel{12}{[-]}$, leaving the superposition $\frac{1}{\sqrt{2}}(\stackrel{03}{(+i)} \pm \stackrel{03}{[-i]})$ unchanged (up to a factor $\pm\frac{i}{2}$). This explains that in the nonrelativistic limit the state $\frac{1}{\sqrt{2}}(\stackrel{03}{(+i)} + \stackrel{03}{[-i]})\stackrel{12}{(+)}$ can be denoted just by the eigenvalue of S^{12}, which is $\frac{1}{2}$, while the state $\frac{1}{\sqrt{2}}(\stackrel{03}{(+i)} + \stackrel{03}{[-i]})\stackrel{12}{[-]}$ is denoted correspondingly by $-\frac{1}{2}$. In the nonrelativistic limit we mostly work with spinors of positive parity.

γ^a's (1))

$$\overset{ab}{(k)}\overset{ab}{(k)} = 0, \quad \overset{ab}{(k)}\overset{ab}{(-k)} = \eta^{aa}\overset{ab}{[k]}, \quad \overset{ab}{[k]}\overset{ab}{[k]} = \overset{ab}{[k]}, \quad \overset{ab}{[k]}\overset{ab}{[-k]} = 0,$$
$$\overset{ab}{(k)}\overset{ab}{[k]} = 0, \quad \overset{ab}{[k]}\overset{ab}{(k)} = \overset{ab}{(k)}, \quad \overset{ab}{(k)}\overset{ab}{[-k]} = \overset{ab}{(k)}, \quad \overset{ab}{[k]}\overset{ab}{(-k)} = 0. \tag{6}$$

We then find that the operators

$$\tau_L^{\mp} := -\overset{03}{(\pm i)}\overset{12}{(\mp)}, \quad \tau_R^{\mp} := \overset{03}{(\mp i)}\overset{12}{(\mp)} \tag{7}$$

transform a state of a particular handedness (left or right) into a state of the same handedness or annihilate it, while they annihilate the states of an opposite handedness

$$\tau_L^{-}|0\rangle_L = |1\rangle_L, \quad \tau_L^{+}|1\rangle_L = |0\rangle_L,$$
$$\tau_R^{-}|0\rangle_R = |1\rangle_R, \quad \tau_R^{+}|1\rangle_R = |0\rangle_R. \tag{8}$$

All the other applications of $\tau_L^{\mp}$ and $\tau_R^{\mp}$ give zero.

We also find that the operators

$$\tau^{\mp} := \tau_L^{\mp} + \tau_R^{\mp} = -\overset{03}{(\pm i)}\overset{12}{(\mp)} + \overset{03}{(\mp i)}\overset{12}{(\mp)} \tag{9}$$

transform a state of a well defined parity (5) into a state of the same parity or annihilate it

$$\tau^{-}|0\rangle_{\pm} = |1\rangle_{\pm}, \quad \tau^{+}|1\rangle_{\pm} = |0\rangle_{\pm}, \tag{10}$$

while the rest of applications give zero, accordingly $(\tau^+ + \tau^-)|0\rangle_{\pm} = |1\rangle_{\pm}$, $(\tau^+ + \tau^-)|1\rangle_{\pm} = |0\rangle_{\pm}$.

We present the following useful properties of $\tau^{\pm}$, valid for $\tau_L^{\pm}$ and $\tau_R^{\pm}$ as well, so that we shall from now on skip the index L, R,

$$(\tau^{\pm})^2 = 0, \quad (\tau^{\pm})^{\dagger} = \tau^{\mp}, \quad \tau^{+}\tau^{-} = \overset{12}{[+]}, \quad \tau^{-}\tau^{+} = \overset{12}{[-]}, \quad (\tau^{+} + \tau^{-})^2 = I,$$
$$\tau^{+}\overset{12}{[+]} = 0, \quad \tau^{-}\overset{12}{[-]} = 0, \quad \overset{12}{[+]}\tau^{-} = 0, \quad \overset{12}{[-]}\tau^{+} = 0, \tag{11}$$
$$\tau^{+}\overset{12}{[-]} = \tau^{+}, \quad \tau^{-}\overset{12}{[+]} = \tau^{-}, \quad \overset{12}{[+]}\tau^{+} = \tau^{+}, \quad \overset{12}{[-]}\tau^{-} = \tau^{-}.$$

We identify one qubit with one of the two spinor basic states, making use of states with positive parity, or we equivalently use the basis with well defined handedness.

We identify n-qubits with states which are superposition of 2^n products of n one spinor states.

A n-qubit state[3] can be written in the chiral representation as

$$|i_1 i_2 \cdots i_l \cdots i_n\rangle_{\alpha} = \prod_{l=1,n} |i_l\rangle_{\alpha}, \quad \alpha = L, R, \tag{12}$$

[3]Let us point out that each operator γ^a carries the index of a particular spinor (γ_l^a) and so do all the projectors and nilpotents ($\overset{ab}{(k)}_l$, $\overset{12}{[k]}_l$)). Spinors "see each other" only by exchanging gauge bosons or in the effective theories by experiencing the (classical) fields (the electromagnetic fields, for example) or the effective spin-

while in the representation with well defined parity (we made a choice of a positive parity) we similarly have

$$|i_1 i_2 \cdots i_l \cdots i_n\rangle_\pm = \prod_{l=1,n} |i_l\rangle_\pm. \quad (13)$$

Here i_l stands for $|0\rangle_l$ or $|1\rangle_l$—the two basic states of the l-th spinor. All the raising and lowering operators $\tau_l^{\alpha\pm}$, $\alpha = L, R$ or $\tau_l^\pm$ carry the index of the corresponding qubit manifesting that they only apply on the state l (on the l-th spinor), while they do not "see" all the other states. Since they are made out of an even number of the Clifford odd nilpotents, they do not bring any sign when jumping over one-qubit states.

Either in the chiral representation or in the representation with well defined parity the basic states are chosen to be the eigenstates of the operators S_l^{12}. According to (10) any n-qubit state can be written as follows

$$|i_1 i_2 \cdots i_l \cdots i_n\rangle = \prod_{l=1,n} (\tau_l^-)^{i_l} |0\rangle_l, \quad (14)$$

with i_l equal to 0 for the state with the eigenvalue of S_l^{12} equal to $\frac{1}{2}$ or 1 for the state with the eigenvalue of S_l^{12} equal to $-\frac{1}{2}$.

3 Quantum Gates

We define in this section three kinds of quantum gates: the phase gate and the Hadamard's gate, which apply on a particular qubit l and the C-NOT gate, which applies on two qubits, say l and m. All three gates are expressed in terms of projectors and an even number of nilpotents.

i. The phase gate $\mathcal{R}_{\Phi_l}$ is defined as

$$\mathcal{R}_{\Phi_l} = \overset{12}{[+]}_l + e^{i\Phi_l} \overset{12}{[-]}_l \,. \quad (15)$$

Statement *The phase gate $\mathcal{R}_{\Phi_l}$ if applying on $|0_l\rangle$ leaves it in the same state $|0_l\rangle$ without any change, while if applying on $|1_l\rangle$ multiplies this state with $e^{i\Phi}$. This is true for states with well defined parity $|i_l\rangle$ and also for the states with well defined handedness ($|i_l\rangle_L$ and $|i_l\rangle_R$).*

Proof To prove this statement one only has to apply the operator $\mathcal{R}_{\Phi_l}$ on $|i_l\rangle$, $|i_l\rangle_L$ and $|i_l\rangle_R$, with i_l equal to 0 or 1, taking into account equations from Sect. 2. □

spin interactions. All these types of interactions manifest besides in the spin S_l^{12} part of a spinor state ($\overset{12}{(+)}_l$, $\overset{12}{[-]}_l$) also in the S_l^{03} part ($\overset{03}{[-i]}_l$, $\overset{o3}{(+)}_l$), although this change is never pointed out, from the reason discussed in footnote 2, although it might be instructive. Two spinor qubits can coherently (not at the same time, but in a controlled way [8], although the degree of coherence must be and it is carefully studied for the purpose of quantum computers) be changed by a laser beam, for example, even if the two qubits are well separated in space.

ii. The Hadamard's gate $\mathcal{H}_l$ is defined as

$$\mathcal{H}_l = \frac{1}{\sqrt{2}}[\overset{12}{[+]}_l - \overset{12}{[-]}_l - \overset{03}{(+i)}_l\overset{12}{(-)}_l + \overset{03}{(-i)}_l\overset{12}{(-)}_l - \overset{03}{(-i)}_l\overset{12}{(+)}_l + \overset{03}{(+i)}_l\overset{12}{(+)}_l], \tag{16}$$

or equivalently in terms of $\tau^{\pm}$ (9)

$$\mathcal{H}_l = \frac{1}{\sqrt{2}}[\overset{12}{[+]}_l - \overset{12}{[-]}_l + \tau_l^- + \tau_l^+]. \tag{17}$$

Statement *The Hadamard's gate $\mathcal{H}_l$ if applying on $|0_l\rangle$ transforms it to $(\frac{1}{\sqrt{2}}(|0_l\rangle + |1_l\rangle))$, while if applying on $|1_l\rangle$ it transforms the state to $(\frac{1}{\sqrt{2}}(|0_l\rangle - |1_l\rangle))$. This is true for states with well defined parity $|i_l\rangle$ and also for the states in the chiral representation $|i_l\rangle_L$ and $|i_l\rangle_R$.*

Proof To prove this statement one only has to apply the operator $\mathcal{H}_l$ on $|i_l\rangle$, $|i_l\rangle_L$ and $|i_l\rangle_R$, with i_l equal to 0 or 1, taking into account equations from Sect. 2. □

iii. The C-NOT gate $\mathcal{C}_{lm}$ is defined as

$$\mathcal{C}_{lm} = \overset{12}{[+]}_l + \overset{12}{[-]}_l\,[-\overset{03}{(+i)}_m\overset{12}{(-)}_m + \overset{03}{(-i)}_m\overset{12}{(-)}_m - \overset{03}{(-i)}_m\overset{12}{(+)}_m + \overset{03}{(+i)}_m\overset{12}{(+)}_m], \tag{18}$$

or equivalently

$$\mathcal{C}_{lm} = \overset{12}{[+]}_l + \overset{12}{[-]}_l\,[\tau_m^- + \tau_m^+]. \tag{19}$$

Statement *The C-NOT gate $\mathcal{C}_{lm}$ if applying on $|\cdots 0_l \cdots 0_m \cdots\rangle$ transforms it back to the same state, if applying on $|\cdots 0_l \cdots 1_m \cdots\rangle$ transforms it back to the same state. If $\mathcal{C}_{lm}$ applies on $|\cdots 1_l \cdots 0_m \cdots\rangle$ transforms it to $|\cdots 1_l \cdots 1_m \cdots\rangle$, while it transforms the state $|\cdots 1_l \cdots 1_m \cdots\rangle$ to the state $|\cdots 1_l \cdots 0_m \cdots\rangle$.*

Proof To prove this statement one only has to apply the operator $\mathcal{C}_{lm}$ on states $|\cdots i_l \cdots i_m \cdots\rangle$, $|\cdots i_l \cdots i_m \cdots\rangle_L$, $|\cdots i_l \cdots i_m \cdots\rangle_R$, with i_l, i_m equal to 0 or 1, taking into account equations from Sect. 2. □

Statement *When applying $\prod_i^n \mathcal{H}_i$ on the n qubit state with all the qubits in the state $|0_i\rangle$, we get the state $|\psi_0^0\rangle$*

$$|\psi_0^0\rangle = \prod_i^n \mathcal{H}_i|0_i\rangle = \frac{1}{2^{n/2}}\prod_i^n(|0_i\rangle + |1_i\rangle). \tag{20}$$

Proof It is straightforward to prove this statement, if the statement ii. of this section is taken into account. □

4 Useful Properties of Quantum Gates in the Technique Using Nilpotents and Projectors

We present in this section some useful relations.

i. One easily finds, taking into account (15,17,19,9,11), the relation

$$\mathcal{R}_{\Phi_l}\mathcal{H}_l\mathcal{R}_{\vartheta_l}\mathcal{H}_l = \frac{1}{2}\{(\overset{12}{[+]}_l + e^{i\varphi_l}\overset{12}{[-]}_l)(1+e^{i\vartheta_l}) + (\tau_l^+ + e^{i\varphi_l}\tau_l^-)(1-e^{i\vartheta_l})\}, \quad (21)$$

which transforms $|i_l\rangle$ into a general superposition of $|0_l\rangle$ and $|1_l\rangle$

$$\begin{aligned} e^{-i\vartheta_l}\mathcal{R}_{(\varphi_l+\pi/2)}\mathcal{H}_l\mathcal{R}_{2\vartheta_l}\mathcal{H}_l|0_l\rangle &= \cos(\vartheta_l)|0_l\rangle + e^{i\varphi_l}\sin(\vartheta_l)|1_l\rangle, \\ e^{-i(\vartheta_l-\pi/2)}\mathcal{R}_{(\varphi_l-\pi/2)}\mathcal{H}_l\mathcal{R}_{2\vartheta_l}\mathcal{H}_l|1_l\rangle &= \sin(\vartheta_l)|0_l\rangle + e^{i\varphi_l}\cos(\vartheta_l)|1_l\rangle. \end{aligned} \quad (22)$$

ii. Let $|\psi^m\rangle_p$ be a general superposition of p-qubit states $|k\rangle_p = \prod_{l=1,p}(\tau_l^-)^{i_l}|0\rangle_l$, with $i_l = 0, 1$ for a particular choice of i_l so that $|k\rangle_p$ represents any of the 2^p basic p-qubit states: $|\psi^m\rangle_p = \sum_{k=1}^{2^p} \alpha_k^m\, |k\rangle_p$. Then for $\sum_{k=1}^{2^p} \alpha_k^{m\star}\alpha_k^m = 1$ we find that the operator $\hat{O}_p^m$ is an involution

$$\begin{aligned} \hat{O}_p^m &= 2|\psi^m\rangle_p\, {}_p\langle\psi^m| - I, \\ (\hat{O}_p^m)^\dagger &= \hat{O}_p^m, \quad (\hat{O}_p^m)^2 = I. \end{aligned} \quad (23)$$

We also find that any operator $\hat{P}_p^k = I - 2\hat{R}_p^k$ with $\hat{R}_p^k = \prod_{l=1}^p\, |i_l\rangle\,\langle i_l| = |k\rangle_p\, {}_p\langle k|$, with a particular choice of i_l, is also an involution

$$\begin{aligned} \hat{P}_p^k &= I - 2\prod_{l=1}^{p}\, |i_l\rangle\,\langle i_l| = I - 2\hat{R}_p^k, \\ (\hat{P}_p^k)^2 &= (I-2\hat{R}_p^k)^2 = I. \end{aligned} \quad (24)$$

5 Algorithm for Extracting Particular States

Let as define a quantum algorithm, which extracts a particular information out of a data base with n qubits. We assume that the starting state is any superposition of 2^p states, out of which we are extracting a particular state. In the case that the starting state is a superposition of 2^p states with equal coefficients (20), all of them equal to $2^{-p/2}$, the algorithm is known as the Grover's algorithm [3]. Let $|k\rangle_p$ be a state of p qubits

$${}_p\langle k|m\rangle_p = \delta^{km}, \quad (25)$$

and let $|k_0\rangle_p$ be a particular state of p qubits, which we would like to extract out of an general superposition $|\psi^m\rangle$ of 2^p orthogonal states $|k\rangle_p$

$$\begin{aligned} |\psi^m\rangle &= \sum_{k=1}^{2^p} \frac{\alpha_k^m}{\sqrt{\sum_{k'}\alpha_{k'}^*\alpha_{k'}}}|k\rangle_p, \\ &= A^m|k_0^m \perp\rangle_p + B^m|k_0\rangle_p, \end{aligned} \quad (26)$$

with

$$
\begin{aligned}
&{}_p\langle k_0|k_0^m \perp\rangle_p = 0, \quad I = \sum_k |\psi^k\rangle_{p\;p}\langle\psi^k|,\\
&|k_0^m \perp\rangle_p = \sum_{k\neq k_0} \frac{\alpha_k^m}{\sqrt{\sum_{k'\neq k_0}\alpha_{k'}^{m\star}\alpha_{k'}^m}}\,|k\rangle_p,\\
&A^m = \cos\vartheta_m = \frac{\sqrt{\sum_{k'\neq k_0}\alpha_{k'}^{m\star}\alpha_{k'}^m}}{\sqrt{\sum_{k'}\alpha_{k'}^{m\star}\alpha_{k'}^m}} = \sqrt{1-\frac{\alpha_{k0}^{m\star}\,\alpha_{k0}^m}{\sum_{k'}\alpha_{k'}^{m\star}\alpha_{k'}^m}},\\
&B^m = \sin\vartheta_m e^{i\varphi_m} = \frac{\alpha_{k_0}^m}{\sqrt{\sum_{k'}\alpha_{k'}^{m\star}\alpha_{k'}^m}}.
\end{aligned}
\tag{27}
$$

Let us define the unitary operator $\hat{\mathcal{E}}_p^m$ (one can easily see—the technique is very useful for this purpose—that there are not additional useful possibilities for a choice of $\hat{\mathcal{E}}_p^m$)

$$
\hat{\mathcal{E}}_p^m = (2|\psi^m\rangle_{p\;p}\langle\psi^m| - I)\,(I - 2|k_0\rangle_{p\;p}\langle k_0|). \tag{28}
$$

We find $\hat{\mathcal{E}}_p^{m\dagger}\hat{\mathcal{E}}_p^m = I$. Since we can write

$$
\begin{aligned}
|\psi^m\rangle_{p\;p}\langle\psi^m| &= \cos^2\vartheta_m |k_0^m \perp\rangle_{p\;p}\langle k_0^m \perp| + \sin^2\vartheta_m |k_0\rangle_{p\;p}\langle k_0|\\
&\quad + \sin\vartheta_m\cos\vartheta_m (e^{-i\varphi_m}|k_0^m \perp\rangle_{p\;p}\langle k_0| + e^{i\varphi_m}|k_0\rangle_{p\;p}\langle k_0^m \perp|),
\end{aligned}
\tag{29}
$$

it accordingly follows

$$
\begin{aligned}
\hat{\mathcal{E}}_p^m\,|\psi^m\rangle_p &= [\cos 2\vartheta_m + \sin 2\vartheta_m (e^{i\varphi_m}\,|k_0\rangle_{p\;p}\langle k_0^m \perp| - e^{-i\varphi_m}|k_0^m \perp\rangle_{p\;p}\langle k_0|)]\,|\psi^m\rangle_p\\
&= \cos(3\vartheta_m)\,|k_0^m \perp\rangle_p + e^{i\varphi_m}\,\sin(3\vartheta_m)\,|k_0\rangle_p.
\end{aligned}
\tag{30}
$$

If we apply the operator $\hat{\mathcal{E}}_p^m$ j times, we find

$$
\begin{aligned}
(\hat{\mathcal{E}}_p^m)^j\,|\psi^m\rangle_p &= [\cos(2j\vartheta_m) + \sin(2j\vartheta_m)(e^{i\varphi_m}\,|k_0\rangle_{pp}\langle k_0^m \perp| - e^{-i\varphi_m}\,|k_0^m \perp\rangle_{pp}\langle k_0|)]\,|\psi^m\rangle_p\\
&= \cos[(2j+1)\vartheta_m]\,|k_0^m \perp\rangle_p + e^{i\varphi_m}\,\sin[(2j+1)\vartheta_m]\,|k_0\rangle_p.
\end{aligned}
\tag{31}
$$

It follows that $(\hat{\mathcal{E}}_p^m)^j$ extracts our particular state $|k_0\rangle_p$ out of the initial state $|\psi^m\rangle_p$, if we choose j so that

$$
[(1+2j)\vartheta_m + \varepsilon] = \frac{\pi}{2} \tag{32}
$$

for as small $|\varepsilon|$ as possible. If we choose the initial state $|\psi^m\rangle_p$ to be just our desired state, then $\vartheta_m = \pi/2$ and $j = 0$. If the initial state has all the coefficients equal to $2^{-p/2}$, then this is the Grover's algorithm [3], provided that $\sin\vartheta_m = 2^{-p/2}$. Not necessarily the generalized algorithm is more efficient than the Grover's one. It might even be, that it is worse if we do not choose ϑ_m equal to $2^{-p/2}$, but when we have some particular knowledge about the desired state, it might be very useful.

6 Concluding Remarks

We presented in this paper how can the Clifford algebra technique [1, 2] be used in quantum computers for generating quantum gates, demonstrating that the Clifford algebra technique [1, 2] makes the formation of quantum gates very transparent and accordingly very simple, for either nonrelativistic or relativistic spinors. Although our projectors and nilpotents can as well be expressed in terms of the ordinary projectors and the ordinary operators (we chose the basic states so that when going into the matrix representation the usually used matrices are reproduced), the elegance of the technique might be very helpful to better understand the operators appearing in the quantum gates and quantum algorithms and to faster see consequences of various applications of the gates.

We present also the algorithm appropriate for extracting an information out of any superposition of 2^n n qubit states. This is a generalization of the well known Grover's algorithm, allowing in principle a faster extraction of a particular information than the Grover's algorithm, if the initial state favours the particular state. Our algorithm becomes the Grover's algorithm for a very special choice of the initial state, out of which the information is extracting.

The presented Clifford algebra technique (with nilpotents and projectors [1, 2]) used in this paper for quantum gates and algorithms for the two states quantum bits can be quite easily generalized to cases with more than two states qubits (qudits). Transparency of our way of presenting quantum gates and algorithms may qualify the technique for further use when searching for new algorithms. A different point of view on any problem may many a time help a lot.

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